National Championship 2020

Premier Division Solutions

April 26 — May 9, 2020

Time Limit: 3 hours.
Each problem is worth 7 points.

1. Let $\mathcal{P}$ be a finite set of squares on an infinite chessboard. Kelvin the Frog notes that $\mathcal{P}$ may be tiled with only 1×2 dominoes, while Alex the Kat notes that $\mathcal{P}$ may be tiled with only 2×1 dominoes. The dominoes cannot be rotated in each tiling. Prove that the area of $\mathcal{P}$ is a multiple of 4.

   Proposed by: Ankan Bhattacharya

   With an arbitrary square set as the origin, color each square in $\mathcal{P}$ as follows:

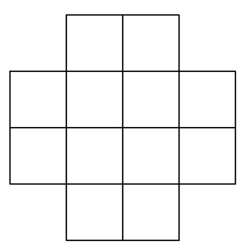
   - If both coordinates are even, color the square red.
   - If the first coordinate is even and the second is odd, color the square yellow.
   - If the first coordinate is odd and the second is even, color the square green.
   - If both coordinates are odd, color the square blue.

   Note that every 1×2 domino covers either one red square and one green square, or one yellow square and one blue square. Thus, the number of red squares is equal to the number of green squares, and the number of yellow squares is equal to the number of blue squares.

   Similarly, every 2×1 domino covers either one red square and one yellow square, or one green square and one blue square. Thus, the number of red squares is equal to the number of yellow squares, and the number of green squares is equal to the number of blue squares.

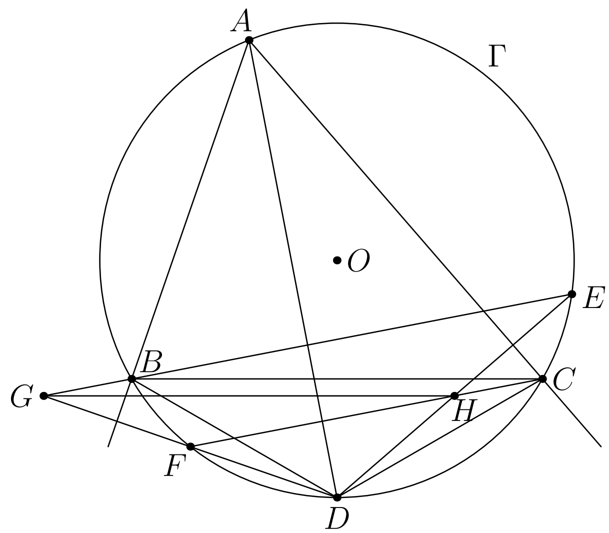
   As a result, each color is used on the same number of squares, implying that the number of squares is a multiple of 4.

   Remark. It is not true that the polygon is a disjoint union of 2×2 squares. As a counterexample, consider the shape below.



2. Let ABC be an acute triangle with circumcircle Γ and let D be the midpoint of minor arc BC. Let E, F be on Γ such that $DE \perp AC$ and $DF \perp AB$. Lines BE and DF meet at G, and lines CF and DE meet at H. Show that $BCHG$ is a parallelogram.

   Proposed by: Kevin Ren



Note that

$$\angle(AD, BE) = \frac{\widehat{BD} + \widehat{AE}}{2} = \frac{\widehat{CD} + \widehat{AE}}{2} = \angle(AC, DE) = 90^\circ.$$

Therefore, $BE \perp AD$. Also $DF \perp AB$, so G is the orthocenter of ABD. Similarly, H is the orthocenter of ACD.

Let O be the circumcenter of ABC, and identify O with the zero vector. Then, $\vec{G} = \vec{A} + \vec{B} + \vec{D}$ and $\vec{H} = \vec{A} + \vec{C} + \vec{D}$, so $\vec{G} - \vec{H} = \vec{B} - \vec{C}$, which means $BCHG$ is a parallelogram.

3. Call a polynomial f with positive integer coefficients *triangle-compatible* if any three coefficients of f satisfy the triangle inequality. For instance, $3x^3 + 4x^2 + 6x + 5$ is triangle-compatible, but $3x^3 + 3x^2 + 6x + 5$ is not. Given that f is a degree 20 triangle-compatible polynomial with -20 as a root, what is the least possible value of $f(1)$?

   Proposed by: Kevin Ren

   Answer: $\boxed{4641}$

   First, consider

   $$f(x) = 11\left(x^{20} + 21x^{19} + 21x^{18} + \cdots + 21x + 21\right) + x + 9.$$

   Note that f is triangle-compatible and $f(-20) = 0$, as required. This f achieves $f(1) = 4641$. Let us show this is optimal.

   Suppose f be triangle compatible. Divide f by the leading coefficient to get g, a monic polynomial with rational coefficients. Write

   $$g(x) = x^{20} + a_{19}x^{19} + a_{18}x^{18} + \cdots + a_1x + a_0.$$

   Let $a_k = b_k + 21$ for $k = 0, 1, \ldots, 19$. By the Triangle Inequality, $|a_i - a_j| = |b_i - b_j| < 1$ for all i, j. Then,

   $$g(x) = \left[x^{20} + 21x^{19} + \cdots + 21x + 20\right] + 1 + \left[b_{19}x^{19} + b_{18}x^{18} + \cdots + b_1x + b_0\right].$$

   The first term has -20 as a root, so $g(-20) = 0$ is equivalent to

   $$1 = 20^{19}b_{19} - 20^{18}b_{18} + \cdots + 20b_1 - b_0. \tag{1}$$

   Let k be the smallest number such that $c_i = kb_i$ is an integer for all $i = 0, \ldots, 19$. $|b_i - b_j| < 1$ implies $|c_i - c_j| \le k - 1$ for all i, j.

   Lemma 1. *If $k \le 11$, then $c_2 = c_3 = \cdots = c_{19} = 0$.*

Proof. Let i be the largest index such that $c_i \neq 0$. Note that

$$k = 20^{19}c_{19} - 20^{18}c_{18} + \cdots + 20c_1 - c_0,$$

and $|c_j| \leq |c_i| + k - 1$ for all $j < i$, while $c_j = 0$ for all $j > i$. Therefore,

$$20^i|c_i| \leq 20^{i-1}|c_{i-1}| + \cdots + 20|c_1| + |c_0| + k \leq \left(20^{i-1} + \cdots + 20 + 1\right)(|c_i| + k - 1) + k.$$

This rearranges to

$$|c_i| \leq \frac{(k-1)\left(20^i + 18\right) + 19}{(18 \cdot 20^i + 1)} \leq \frac{10\left(20^i + 18\right) + 19}{(18 \cdot 20^i + 1)}.$$

For $i \geq 2$, the right-hand side is less than 1, contradiction. □

When $k \leq 11$, Lemma 1 implies $c_2 = c_3 = \cdots = c_{19} = 0$, so $k = 20c_1 - c_0$. By the Triangle Inequality, $|c_0| = |c_0 - c_{19}| \leq k - 1$, so $20|c_1| = |k + c_0| \leq 2k - 1$. For $k \leq 10$, this implies $c_1 = 0$, so $c_0 = -k$, which is a contradiction. So, there are no solutions with $k \leq 10$. If $k = 11$, the only solution is $c_0 = 9, c_1 = 1$, which corresponds to the construction above.

We finish by showing $k \geq 12$ does worse.

Lemma 2. $b_{19} > -\frac{1}{2}$.

Proof. If $b_{19} \leq -\frac{1}{2}$, $|b_i - b_{19}| < 1$ implies $-\frac{3}{2} < b_i < \frac{1}{2}$ for each $0 \leq i \leq 18$. Then, (1) implies

$$\frac{1}{2} \cdot 20^{19} \leq 20^{19}|b_{19}| \leq 20^{18}|b_{18}| + 20^{17}|b_{17}| + \cdots + 20^1|b_1| + |b_0| + 1 < \frac{3}{2}\left(20^{18} + \cdots + 20^1 + 1\right) + 1.$$

This is a contradiction. □

Thus, $b_{19} \geq -\frac{1}{2}$, so $b_i \geq -\frac{3}{2}$ for each $0 \leq i \leq 18$. If $k \geq 12$, then

$$P(1) \geq 12\left(1 + \left(21 - \frac{1}{2}\right) + 19\left(21 - \frac{3}{2}\right)\right) = 4704 > 4641.$$

Therefore, 4641 is minimal.

4. Suppose $n > 1$ is an odd integer satisfying $n \mid 2^{\frac{n-1}{2}} + 1$. Prove or disprove that n is prime.

   Note: unfortunately, the original form of this problem did not include the red text, rendering it unsolvable. We sincerely apologize for this error and are taking concrete steps to prevent similar issues from reoccurring, including computer-verifying problems where possible. All teams will receive full credit for the question.

   Proposed by: Kevin Ren

   We claim that $n = 3277 = 29 \cdot 113$ is a counterexample to the question. We first compute

$$k = \frac{29 \cdot 113 - 1}{2} \equiv \frac{112 + 14 \cdot 226}{2} \equiv 56 + 7 \cdot 2 \equiv 70 \pmod{224}. \tag{2}$$

   In particular, $k \equiv 14 \pmod{56}$. Using Fermat's Little Theorem, we see that

$$2^k \equiv 2^{14} \equiv 3^2 \cdot 2^4 \equiv -1 \pmod{29} \tag{3}$$

   and

$$2^k \equiv 2^{70} \equiv 15^{10} \equiv (-1)^5 \equiv -1 \pmod{113} \tag{4}$$

   Thus, $29 \cdot 113 \mid 2^{\frac{29 \cdot 113 - 1}{2}} + 1$.

5. Alex the Kat and Kelvin the Frog play a game on a complete graph with n vertices. Kelvin goes first, and the players take turns selecting either a single edge to remove from the graph, or a single vertex to remove from the graph. Removing a vertex also removes all edges incident to that vertex. The player who removes the final vertex wins the game. Assuming both players play perfectly, for which positive integers n does Kelvin have a winning strategy?

   Proposed by: Alexander Katz

   Answer: $\boxed{\text{All non-multiples of 3}}$

   Say a graph G is *winning* if the first player to move has a winning strategy, and *losing* otherwise.

   Say vertices $x, y \in G$ are a *symmetric pair* if the neighbor sets of x and y are the same (and in particular, x and y are not adjacent). Let $f_{x,y}(G)$ denote the graph obtained from G by deleting x and y, and all edges incident to them.

   Lemma 1. *If vertices x, y are a symmetric pair of graph G, then G is winning if and only if $f_{x,y}(G)$ is winning.*

   Proof. We proceed by induction. Suppose the lemma is true for any subgraph of G. We define the notation $G - m$, where m is a vertex or edge, to mean the graph produced by deleting m from G.

   Suppose $f_{x,y}(G)$ is winning. Then, $f_{x,y}(G) - m$ is a losing graph for some $m \in f_{x,y}(G)$. The first player can remove the same vertex or edge from G. Note that x, y is still a symmetric pair of $G - m$, and $f_{x,y}(G) - m = f_{x,y}(G - m)$. By induction, $G - m$ is losing, so G is winning.

   Conversely, suppose $f_{x,y}(G)$ is losing. Consider any move $m \in G$ by the first player.

   Suppose further that $m \in f_{x,y}(G)$. Since $f_{x,y}(G)$ is losing, $f_{x,y}(G) - m$ is winning. Moreover, x, y is still a symmetric pair of $G - m$, and $f_{x,y}(G - m) = f_{x,y}(G) - m$. By induction $G - m$ is winning.

   Otherwise, $m \notin f_{x,y}(G)$, so m is either vertex x or y, or one of the edges from x or y to $f_{x,y}(G)$. The second player will answer with move m', defined as follows: if the first player removed an edge (x, v) (resp. (y, v)), the second player removes edge (y, v) (resp. (x, v)), and if the first player removed x (resp. y), the second player removes y (resp. x). By induction, the new graph $G - m - m'$ is losing because $f_{x,y}(G)$ is losing. Since a move m' exists for each $m \notin f_{x,y}(G)$, $G - m$ is winning.

   Therefore, $G - m$ is winning for any $m \in G$, so G is losing. $\square$

   We claim that Kelvin wins precisely when n is not a multiple of 3. This is evident for $n = 1, 2, 3$, so we proceed again by induction, assuming the result for $k < n$.

   If $n \equiv 1 \pmod 3$, Kelvin removes any vertex. This leaves Alex with a complete graph on $n - 1 \equiv 0 \pmod 3$ vertices, which by induction is losing. Therefore, Kelvin wins.

   If $n \equiv 2 \pmod 3$, Kelvin removes any edge (x, y). In the resulting graph $G = K_n - (x, y)$, x, y is a symmetric pair. Note that $f_{x,y}(G) = K_{n-2}$ is losing because $n - 2 \equiv 0 \pmod 3$. By Lemma 1, G is also losing. Therefore, Kelvin wins.

   If $n \equiv 0 \pmod 3$, we will show Kelvin loses. If Kelvin removes any vertex, Alex gets a complete graph on $n - 1 \equiv 2 \pmod 3$ vertices, which by induction is winning. If Kelvin removes any edge (x, y), Alex gets a graph $G = K_n - (x, y)$, where x, y is a symmetric pair. By induction, $f_{x,y}(G) = K_{n-2}$ is winning because $n - 2 \equiv 1 \pmod 3$, so G is winning by Lemma 1. Since any move leads to a winning position for Alex, Kelvin loses.

6. Let P be a non-constant polynomial with integer coefficients such that if n is a perfect power, so is $P(n)$. Prove that $P(x) = x$ or P is a perfect power of a polynomial with integer coefficients.

   A perfect power is an integer n^k, where $n \in \mathbb{Z}$ and $k \geq 2$. A perfect power of a polynomial is a polynomial $P(x)^k$, where P has integer coefficients and $k \geq 2$.

   Proposed by: Kevin Ren

   We write $f = a f_0^{e_0} f_1^{e_1} f_2^{e_2} \cdots f_k^{e_k}$, where a is an integer, $f_0 \equiv x$ and for all $i = 1, \ldots, k$, f_i is a non-constant irreducible polynomial with $f_i(0) \neq 0$ and e_i is a positive integer. (Note that we may have $e_0 = 0$.)

Lemma 1. *If g and h are relatively prime polynomials, then there are finitely many primes p such that there exists a positive integer x satisfying $p \mid \gcd(g(x), h(x))$.*

Proof. By Bézout's Lemma, we can find integer polynomials $A(x), B(x)$ such that

$$A(x)g(x) + B(x)h(x) = N$$

for some integer N. So, $p \mid N$. $\square$

Lemma 2. *If g is non-constant and irreducible with $g(0) \neq 0$, then there exist infinitely many primes p such that for some positive integer x_p, $p \nmid x_p$ and $v_p(g(x_p)) = 1$.*

Proof. Since g is irreducible, g is relatively prime to its derivative g'. Since g is non-constant, there exist infinitely many primes p such that for some positive integer y_p, $p \mid g(y_p)$. Since g, g' are relatively prime, by Lemma 1 there are only finitely many p for which we simultaneously have $p \mid g'(y_p)$. By discarding these and the divisors of $g(0)$, we get infinitely many primes p such that $p \mid g(y_p)$, $p \nmid g'(y_p)$ and $p \nmid g(0)$. If $p \mid y_p$, then $p \mid g(y_p)$ implies $p \mid g(0)$; therefore $p \nmid y_p$. For each of these p,

$$g(y_p + p) - g(y_p) \equiv pg'(y_p) \pmod{p^2}.$$

So, if $v_p(g(y_p)) = 1$, we can take $x_p = y_p$, and if $v_p(g(y_p)) \geq 2$, then $v_p(g(x_p + p)) = 1$ and we can take $x_p = y_p + p$. $\square$

First suppose $k \geq 1$. We claim that we can pick distinct primes p_i and integers x_i, for $i = 0, 1, \ldots, k$, such that:

(i) $v_{p_i}(f_i(x_i)) = 1$ for all i;

(ii) $v_{p_i}(f_j(x_i)) = 0$ for all $i \neq j$;

(iii) $p_i \nmid x_i$ for all $i \neq 0$;

(iv) p_i is relatively prime to a for all i.

Indeed, Lemma 2 gives infinitely many p_i satisfying (i) and (iii) for each $i > 0$. There are also infinitely many p_i satisfying (i) for $i = 0$: just take p_0 to be any prime and $x_0 = p_0$. By Lemma 1, for each pair (i, j) there are only finitely many primes p for which there exists x satisfying $p | \gcd(f_i(x), f_j(x))$, and by excluding these p we get (ii). We get (iv) by excluding only finitely many more primes.

By the Chinese Remainder Theorem, there exists a positive integer x such that $x \equiv x_i \pmod{p_i^2}$. Let $N = \prod_{i=0}^{k} [p_i(p_i - 1)e_i]$, and let $t = x^{N+1}$. For each $i = 0, 1, \ldots, k$,

$$v_{p_i}\left(f(t)\right) = e_0(1 + N)v_{p_i}(x) + \sum_{j=1}^{k} e_j v_{p_i}\left(f_j\left(x^{N+1}\right)\right),$$

where we use that $v_{p_i}(a) = 0$.

Consider first $i > 0$. Since $p_i \nmid x_i$, $x \equiv x_i \not\equiv 0 \pmod{p_i}$. Thus $v_{p_i}(x) = 0$. For each $j > 0$ with $j \neq i$, note that $x^{N+1} \equiv x \pmod{p_i}$ by Fermat's Little Theorem, using that $p_i - 1 | N$. Therefore,

$$f_j(x^{N+1}) \equiv f_j(x) \equiv f_j(x_i) \not\equiv 0 \pmod{p_i},$$

and so $v_{p_i}\left(f_j\left(x^{N+1}\right)\right) = 0$. Moreover, writing $N = p_i N'$ where $p_i - 1 | N'$, we have

$$v_{p_i}\left(x^{N+1} - x\right) = v_{p_i}\left(x^{p_i N'} - 1\right) = v_{p_i}\left(x^{N'} - 1\right) + 1 \geq 2.$$

So,

$$f_i\left(x^{N+1}\right) \equiv f_i\left(x\right) \equiv f_i(x_i) \pmod{p_i^2},$$

and so $v_{p_i}\left(f_i\left(x^{N+1}\right)\right) = 1$. Putting this together, we get $v_{p_i}(f(t)) = e_i$.

If $i = 0$, then $v_{p_0}(x) = 1$, while $v_{p_0}\left(f_j\left(x^{N+1}\right)\right) = 0$ for $j > 0$ by the same argument as above. So, $v_{p_0}(f(t)) = e_0(1+N)$.

If $f(t)$ is a perfect gth power, then $g \mid v_{p_i}(f(t))$ for each $i = 0, 1, \ldots, k$, so

$$g \mid \gcd(e_0(1+N), e_1, \ldots, e_k) = \gcd(e_0, e_1, \ldots, e_k),$$

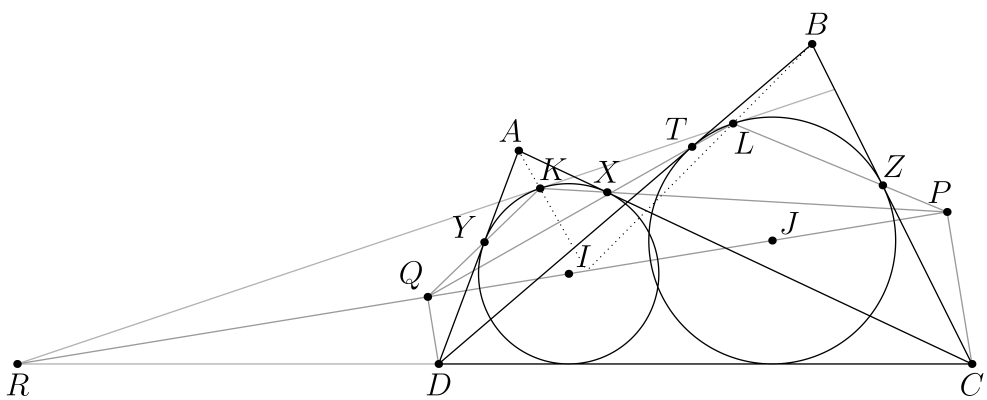
where we use $e_i \mid N$. So, $\frac{f(t)}{a} = f_0(t)^{e_0} f_1(t)^{e_1} \cdots f_k(t)^{e_k}$ is a perfect gth power. Thus, a is a perfect gth power. So, f is a perfect gth power of a polynomial.

Next, if $k = 0$, then $f(x) = ax^n$ for some n. If $a = 1$ then $f(x) = x^n$, and we are done. Otherwise, choose large primes $p \nmid a$ and q such that a is not a qth power; then $f(p^q) = ap^{qn}$ is a perfect gth power for some $g > 1$. So, a and p^{qn} are both gth powers. This gives $g \mid qn$. However, a is not a qth power, so in fact $g|n$. Thus f is a gth power of a polynomial.

7. Let $ABCD$ be a convex quadrilateral, and let ω_A and ω_B be the incircles of $\triangle ACD$ and $\triangle BCD$, with centers I and J. The second common external tangent to ω_A and ω_B touches ω_A at K and ω_B at L. Prove that lines AK, BL, IJ are concurrent.

Proposed by: Ankan Bhattacharya

Solution 1, by Nikolai Beluhov. Let ω_A touch $\overline{AC}$, $\overline{AD}$ at X, Y, and let ω_B touch $\overline{BC}$, $\overline{BD}$ at Z, T. By the Iran Lemma, $P = \overline{KX} \cap \overline{LZ}$ is the foot from C to $\overline{IJ}$, and $Q = \overline{KY} \cap \overline{LT}$ is the foot from D to $\overline{IJ}$. In addition, define $R = \overline{CD} \cap \overline{IJ} \cap \overline{KL}$.



To show that $\overline{AK} \cap \overline{BL}$ lies on $\overline{PQR}$, it suffices to prove $(\overline{KP}, \overline{KQ}; \overline{KR}, \overline{KA}) = (\overline{LP}, \overline{LQ}; \overline{LR}, \overline{LB})$. In fact, we show both are equal to -1:

Lemma 1. $(\overline{KP}, \overline{KQ}; \overline{KR}, \overline{KA}) = -1$.

Proof. Projecting onto ω_A, the cross-ratio equals $(XY;KS)$, where S is the second intersection of line AK with ω_A. Thus $(XY, KS) = -1$ as desired. $\square$

Solution 2, by Ankan Bhattacharya. Let X be the point on line IJ such that $\overline{IJ}$ (externally) bisects $\angle CXD$. We prove that X is the concurrency point. Indeed, the claim that A, K, X collinear reduces to the following (with relabeled point names):

Lemma 2. *Let the incircle of $\triangle ABC$ have center I and touch $\overline{BC}$ at D. Let X be a point so that $\overline{XI}$ externally bisects $\angle BXC$. Then $\overline{XI}$ also (internally) bisects $\angle AXD$.*

Proof. Apply Dual Desargues Involution Theorem from X on the (degenerate) complete quadrilateral $\{\overline{AB}, \overline{BD}, \overline{DC}, \overline{CA}\}$. $\square$

Remark (Nikolai Beluhov). The above lemma may be proved without DDIT.

8. Let n, m be positive integers, and let α be an irrational number satisfying $1 < \alpha < n$. Define the set

$$X = \{a + b\alpha : 0 \le a \le n \text{ and } 0 \le b \le m\}.$$

Let $x_0 \le x_1 \le \cdots \le x_{(n+1)(m+1)-1}$ be the elements of X. Show that for all $i + j \le (n+1)(m+1) - 1$, we have that $x_{i+j} \le x_i + x_j$.

Proposed by: Ryan Alweiss and Yang Liu

Write $x_k = a_k + b_k\alpha$ where $0 \le a_k \le n$ and $0 \le b_k \le m$. Throughout the proof, assume that $x_i + x_j \le n + m\alpha$ or else the result is trivial.

We will show the following.

Lemma 1. *There are i elements of X in the range $(x_j, x_i + x_j]$.*

It is direct to see that Lemma 1 implies our desired conclusion. We now describe how to show it.

We will construct an injection $f : [i] \to X \cap (x_j, x_i + x_j]$, i.e. we explicitly construct i elements of X in the range $(x_j, x_i + x_j]$. Below we describe the injection. Here, $[i] = \{1, 2, \cdots, i\}$.

$f(k)$ is defined as follows.

(a) If $a_j + a_k \le n$ and $b_j + b_k \le m$ then $f(k) = (a_j + a_k) + (b_j + b_k)\alpha$.

(b) If $b_j + b_k > m$ then $f(k) = (a_j + a_k + \lceil (b_j + b_k - m)\alpha \rceil) + (m - b_k)\alpha$.

(c) If $a_j + a_k > n$ then define $\ell := \left\lceil \frac{a_j + a_k - n}{\alpha} \right\rceil$. Define

$$h_k := \max\left(0, a_j - \lfloor \ell\alpha \rfloor\right) + \left(a_j + a_k - n - 1 - \lfloor (\ell - 1)\alpha \rfloor\right)$$

and then $f(k) = h_k + (b_j + b_k + \ell)\alpha$.

Lemma 2. *If $x_i + x_j \le n + m\alpha$, then the function f as described above sends $[i]$ to $X \cap (x_j, x_i + x_j]$ and is an injection.*

Proof. We split the proof into its pieces.

Part 1. f is well-defined. The only thing to check here is that we cannot have both $a_j + a_k > n$ and $b_j + b_k > m$. Indeed, this would imply that $x_j + x_k > n + m\alpha$, which is absurd.

Part 2. f is an injection. We can write $f(k) = c_k + d_k\alpha$. We will analyze c_k and d_k to show that f is an injection. In the first case where $a_j + a_k \le n$ and $b_j + b_k \le m$ note that $a_j \le c_k \le n$ and $b_j \le d_k \le m$. In the second case where $b_j + b_k > m$ we have that $c_k \ge a_j$ and $d_k = m - b_k < b_j$, hence it does not overlap with the first case. Additionally, we can easily recover (a_k, b_k) from $f(k)$ in the first two cases. In the third case, we can check that $h_k < a_j$, hence does not intersect with the first case or second case.

We now show that $h_k < a_j$. Indeed, we have by the definition of ℓ that $\lfloor \ell\alpha \rfloor \ge a_j + a_k - n$, hence $h_k \le a_j - 1 < a_j$ as desired. We now have to argue that if $h_k = h_{k'}$ then $a_k = a_{k'}$ – this would complete the proof that f is injective. We do casework based on whether the the term $\max(0, a_j - \lfloor \ell\alpha \rfloor)$ in the definition of h_k evaluates to 0 or not. In the case where it evaluates to 0, it suffices to check that

$$(a_j + a_k - n - 1 - \lfloor (\ell - 1)\alpha \rfloor) < a_j - \lfloor (\ell - 1)\alpha \rfloor$$

which is equivalent to $a_k < n + 1$, which is true. In the other case, we check that

$$a_j - \lfloor \ell\alpha \rfloor + (a_j + a_k - n - 1 - \lfloor (\ell - 1)\alpha \rfloor) < a_j - \lfloor (\ell - 1)\alpha \rfloor$$

which is equivalent to $a_j + a_k - n - 1 - \lfloor \ell\alpha \rfloor < 0$, which follows from the definition of ℓ.

Part 3. $x_j < f(k) \leq x_j + x_k$ for all $k \in [i]$. In the first case, it is obvious that $x_j < f(k) \leq x_j + x_k$. In the second case, we can compute that

$$f(k) < a_j + a_k + (b_j + b_k - m)\alpha + 1 + (m - b_k)\alpha \leq a_j + b_j\alpha + a_k + b_k\alpha = x_j + x_k,$$

where we have used $b_k \geq 1$. Also, we have that

$$f(k) > a_j + a_k + (b_j + b_k - m)\alpha + (m - b_k)\alpha > a_j + b_j\alpha = x_j.$$

Now we analyze the third case. The case where $\max(0, a_j - \lfloor \ell\alpha \rfloor) = a_j - \lfloor \ell\alpha \rfloor$ follows similarly to the previous case. Let us analyze the case where the max is 0.

Then

$$f(k) = h_k + (b_j + b_k + \ell)\alpha = x_j + x_k + (\ell\alpha - \lfloor (\ell - 1)\alpha \rfloor - (n+1) \leq x_j + x_k + (\alpha + 1 - (n+1)) \leq x_j + x_k$$

by the assumption $1 < \alpha < n$. Also, we have that

$$f(k) \geq a_j - \lfloor \ell\alpha \rfloor + (a_j + a_k - n - 1 - \lfloor (\ell - 1)\alpha \rfloor) + (b_j + b_k + \ell)\alpha > x_j + (a_j + a_k - n - 1 - \lfloor (\ell - 1)\alpha \rfloor.$$

We claim the right quantity is ≥ 0. Indeed, we have that

$$a_j + a_k - n > (\ell - 1)\alpha$$

by the definition of α, hence

$$(a_j + a_k - n - 1 - \lfloor (\ell - 1)\alpha \rfloor > -1.$$

Now, it is ≥ 0 as it is an integer.

Part 4. $f(k) \in X$. It suffices to check that if we write $f(k) = c_k + d_k\alpha$ then $0 \leq c_k \leq n$ and $0 \leq d_k \leq m$. This is trivial to check for the first case. For the second case, note that

$$c_k = \lceil a_j + a_k + (b_j + b_k - m)\alpha \rceil = \lceil x_j + x_k - m\alpha \rceil \leq \lceil n + m\alpha - m\alpha \rceil = n$$

as desired. For the third case, note that $d_k = b_j + b_k + \ell$. We have that

$$d_k\alpha = (b_j + b_k)\alpha + \ell\alpha < (b_j + b_k)\alpha + (a_j + a_k - n) + \alpha < (m+1)\alpha$$

by the definition of ℓ. Therefore, $d_k < m + 1$, so $d_k \leq m$ as desired. $\square$

Remark. The problem actually holds for $1 < \alpha < n+1$. This needs one more detail. The only time the upper bound enters into the computation is in the third paragraph showing $f(k) \leq x_j + x_k$. If $\alpha > n$ then $\ell = 1$ always, so it reduces to $\alpha - (n+1) < 0$, which is true.