USMCA National Championship 2020

Challenger / Open Division Solutions

April 26 — May 9, 2020

Time Limit: 3 hours for all teams in the Challenger Division, and for teams in the Open Division with more than four members. 4 hours for teams in the Open Division with four or fewer members.
Each problem is worth 1 point.

1. If U, S, M, C, A are distinct (not necessarily positive) integers such that $U \cdot S \cdot M \cdot C \cdot A = 2020$, what is the greatest possible value of $U + S + M + C + A$?

   Proposed by: Kevin Ren

   Answer: $\boxed{505}$

   We want to make one of U, S, M, C, A as large as possible. WLOG let the largest of these numbers be U. If $U = 2020$ or $U = 1010$, the remaining four factors multiply to 1 or 2, which is not possible. The next largest factor of 2020 is $U = 505$, which gives the unique solution $2020 = 505 \cdot 2 \cdot 1 \cdot (-1) \cdot (-2)$, and a sum of 505.

   Let us show this is optimal. The next largest factor of 2020 is $U = 404$. We claim that for $U \leq 404$, $U + S + M + C + A < 505$ even when S, M, C, A are not required to be distinct. Then, $S + M + C + A$ is clearly maximized when S, M, C, A are $\frac{2020}{U}, 1, 1, 1$, and $U + S + M + C + A = U + \frac{2020}{U} + 3$, where $U \geq \frac{2020}{U}$. This is maximized at $U = 404$, where the sum is $412 < 505$, as desired.

2. Sarah is fighting a dragon in DnD. She rolls two fair twenty-sided dice numbered $1, 2, \ldots, 20$. She vanquishes the dragon if the product of her two rolls is a multiple of 4. What is the probability that the dragon is vanquished?

   Proposed by: Kevin Ren

   Answer: $\boxed{\frac{1}{2}}$

   Sarah vanquishes the dragon if both rolls are even, which occurs with probability $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, or if one is a multiple of 4 and the other is odd, which occurs with probability $2 \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{4}$.

   So, the total probability is $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

3. If $x(y + 1) = 41$ and $x^2(y^2 + 1) = 881$, determine all possible pairs of real numbers (x, y).

   Proposed by: Brice Huang

   Answer: $\boxed{\left(16, \frac{25}{16}\right), \left(25, \frac{16}{25}\right)}$

   Let $z = xy$. The given equations imply $y + z = 41$ and $y^2 + z^2 = 881$. Thus,

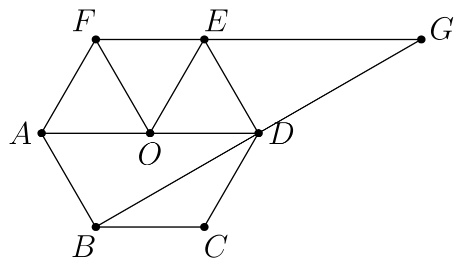
   $$yz = \frac{(y + z)^2 - (y^2 + z^2)}{2} = 400.$$

   By Vieta's Formulas, y, z are the zeros of the quadratic $t^2 - 41t + 400 = 0$.

   This quadratic factors as $(t - 16)(t - 25)$. So, $(y, z) = (16, 25), (25, 16)$. As $z = xy$ the solutions are $(x, y) = \left(16, \frac{25}{16}\right), \left(25, \frac{16}{25}\right)$.

4. Let $ABCDEF$ be a regular hexagon with side length two. Extend FE and BD to meet at G. Compute the area of $ABGF$.

   Answer: $\boxed{7\sqrt{3}}$



Draw segments from A, D, E, F to the center O of the hexagon. Triangles DOE, EOF, FOA are equilateral with side length 2, so their areas are each $\frac{2^2\sqrt{3}}{4} = \sqrt{3}$. Triangle ABD has the same height and twice the base, for an area of $2\sqrt{3}$. Triangle EDG is congruent to triangle ABD (by e.g. $ED = AB$, $\angle DEG = \angle BAD$ and $\angle EDG = \angle ABD$), so its area is also $2\sqrt{3}$. This gives a total area of $7\sqrt{3}$.

5. Call a positive integer n an $A-B$ number if the base A and base B representations of n are three-digit numbers that are reverses of each other. For example, 87 is a $5-6$ number because $87 = 223_6 = 322_5$. Compute the sum of all $7-11$ numbers.

   Proposed by: Kevin Ren

   Answer: $\boxed{437}$

   Let the base 7 and 11 representations be abc_7 and cba_{11}, where $1 \le a, c \le 6$ and $0 \le b \le 6$. Then, $49a + 7b + c = 121c + 11b + a$. This implies $12a = b + 30c$, so b is a multiple of 6. Thus $b = 0$ or $b = 6$.

   If $b = 0$, then $2a = 5c$ implies $a = 5, c = 2$. This gives the solution $502_7 = 205_{11} = 247$. If $b = 6$, then $2a = 1 + 5c$ implies $a = 3, c = 1$. This gives the solution $361_7 = 163_{11} = 190$.

   The answer is $247 + 190 = 437$.

6. Alex is thinking of a number that is divisible by all of the positive integers 1 through 200 inclusive except for two consecutive numbers. What is the smaller of these numbers?

   Proposed by: Aaron Cao

   Answer: $\boxed{127}$

   Let n be Alex's number, and let m be the even non-divisor. If m is not a power of 2, we can write $m = 2^k o$ for $k \ge 1$ and odd $o > 1$. But then n is divisible by 2^k and o, so it must be divisible by m, which is a contradiction. Therefore m is a power of 2.

   $2^7 = 128$ is the largest power of 2 less than 200. If $m < 128$, n is divisible by 128 but not m, which is a contradiction. Therefore $m = 128$.

   The odd non-divisor cannot be 129 because $129 = 3 \cdot 43$. So, the odd non-divisor is 127. As 127 is prime, this pair works.

7. Jenn is competing in a puzzle hunt with six regular puzzles and one additional meta-puzzle. Jenn can solve any puzzle regularly. Additionally, if she has already solved the meta-puzzle, Jenn can also back-solve a puzzle. A back-solve is distinguishable from a regular solve. The meta puzzle cannot be the first puzzle solved. How many possible solve orders for the seven puzzles are possible?

   For example, Jenn may solve #3, solve #5, solve #6, solve the meta-puzzle, solve #2, solve #1, and then solve #4. However, she may not solve #2, solve #4, solve #6, back-solve #1, solve #3, solve #5, and then solve the meta-puzzle.

   Proposed by: Kevin Ren

   Answer: $\boxed{45360}$

   We count the solve orders by first deciding the order the six non-meta puzzles get solved, then deciding where to insert the meta-puzzle in that order, and then deciding which of the puzzles solved after the meta-puzzle were back-solved.

There are 6! solve orders for the non-meta puzzles. The meta-puzzle can be inserted $0, 1, \ldots, 5$ positions from the end; if it is inserted k positions from the end, there are 2^k ways to choose which of the k puzzles solved after the meta-puzzle were back-solved. This gives a count of

$$6!(1+2+4+8+16+32) = 720 \cdot 63 = 45360.$$

8. Two altitudes of a triangle have lengths 8 and 15. How many possible integer lengths are there for the third altitude?

   Proposed by: Kevin Ren

   Answer: $\boxed{12}$

   Let h denote the length of the third altitude, and let A denote the area of the triangle. Then the side lengths of the triangle are $\frac{2A}{8}$, $\frac{2A}{15}$, and $\frac{2A}{h}$. By the Triangle Inequality,

$$\frac{2A}{8} - \frac{2A}{15} < \frac{2A}{h} \implies h \leq 17,$$
$$\frac{2A}{8} + \frac{2A}{15} > \frac{2A}{h} \implies h \geq 6.$$

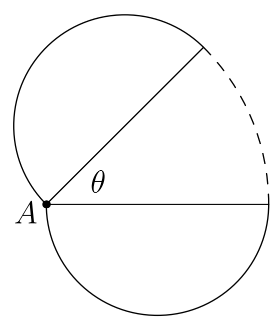
   This gives a count of $17 - 6 + 1 = 12$.

9. Let Ω be a unit circle and A be a point on Ω. An angle $0 < \theta < 180°$ is chosen uniformly at random, and Ω is rotated θ degrees clockwise about A. What is the expected area swept by this rotation?

   Proposed by: Freya Edholm

   Answer: $\boxed{2\pi}$

   The area swept by this rotation consists of two semicircles of radius 1 and a circular sector of radius 2 with measure θ. Thus, its area is equal to $\frac{\pi}{2} + \frac{\pi}{2} + 2\theta$, with θ in radians. The expected value of 2θ is π, so the answer is $\frac{\pi}{2} + \frac{\pi}{2} + \pi = 2\pi$.



10. If $0 < x < \frac{\pi}{2}$ and $\frac{\sin x}{1+\cos x} = \frac{1}{3}$, what is $\frac{\sin 2x}{1+\cos 2x}$?

    Proposed by: Kevin Ren

    Answer: $\boxed{\frac{3}{4}}$

    By the tangent half-angle formula, the condition is $\tan\frac{x}{2} = \frac{1}{3}$. So,

$$\frac{\sin 2x}{1+\cos 2x} = \tan x = \frac{2 \cdot \frac{1}{3}}{1 - \frac{1}{3^2}} = \frac{3}{4}.$$

11. A permutation of $USMCAUSMCA$ is selected uniformly at random. What is the probability that this permutation is exactly one transposition away from $USMCAUSMCA$ (i.e. does not equal $USMCAUSMCA$, but can be turned into $USMCAUSMCA$ by swapping one pair of letters)?

    Answer: $\boxed{\frac{1}{2835}}$

Let us count the number of permutations of $USMCAUSMCA$, and the number of permutations that are one transposition away from $USMCAUSMCA$.

There are $\frac{10!}{2!^5}$ distinct permutations of $USMCAUSMCA$. There are $\frac{10\cdot 8}{2}$ permutations that are one transposition away from $USMCAUSMCA$: there are 10 ways to choose the first letter to swap and 8 ways to choose the letter to swap it with, overcounting for symmetry by a factor of 2.

This gives a probability of

$$\frac{\frac{10\cdot 8}{2}}{\frac{10!}{2!^5}} = \frac{2^4}{9\cdot 7!} = \frac{1}{2835}.$$

12. Let a, b, c, d be the roots of the quartic polynomial $f(x) = x^4 + 2x + 4$. Find the value of

$$\frac{a^2}{a^3+2} + \frac{b^2}{b^3+2} + \frac{c^2}{c^3+2} + \frac{d^2}{d^3+2}.$$

Proposed by: Kevin Ren

Answer: $\boxed{\frac{3}{2}}$

From $a^4 + 2a + 4 = 0$ we get $a^3 + 2 = -\frac{4}{a}$. Thus,

$$\frac{a^2}{a^3+2} = -\frac{a^3}{4} = -\frac{1}{4}\left(-2 - \frac{4}{a}\right) = \frac{1}{2} + \frac{1}{a}$$

and similarly for b, c, d. So the answer is

$$2 + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right).$$

Vieta's Formulas give

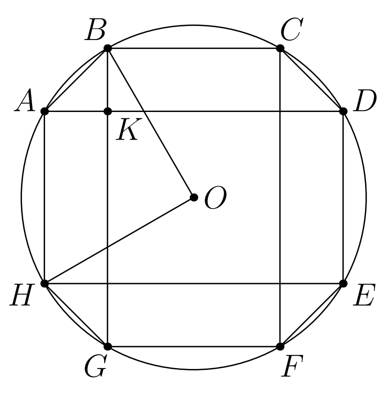
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{abc + abd + acd + bcd}{abcd} = -\frac{1}{2},$$

so the answer is $2 - \frac{1}{2} = \frac{3}{2}$.

13. Equiangular octagon $ABCDEFGH$ is inscribed in a circle centered at O. Chords AD and BG intersect at K. Given that $AB = 2$ and the octagon has area 15, compute the area of $HAKBO$.

Proposed by: Kevin Ren

Answer: $\boxed{\frac{11}{4}}$



By symmetry, $[ABOH] = \frac{1}{4}[ABCDEFGH] = \frac{15}{4}$. Moreover, $[ABK] = 1$ because it is a $45° - 45° - 90°$ triangle with hypotenuse 2. Thus, $[HAKBO] = [ABOH] - [ABK] = \frac{11}{4}$.

14. Kelvin the Nanofrog is visiting his friend, Alex the Nanokat, who lives 483 nanometers away. On his trip to Alex's home, Kelvin travels at k nanometers an hour, where k is an integer, and completes the trip in an integer number of minutes. On his return journey, he travels slower by 7 nanometers an hour, and completes the trip in an integer number of minutes. What is the smallest total number of minutes Kelvin could have spent traveling?

   Proposed by: Brice Huang

   Answer: $\boxed{182}$

   Kelvin's trips take $\frac{483\cdot 60}{k}$ and $\frac{483\cdot 60}{k-7}$ minutes. If these two quantities are integers, so is their difference. Let their difference be α, so

$$\alpha = \frac{483\cdot 60}{k-7} - \frac{483\cdot 60}{k} = \frac{483\cdot 60\cdot 7}{k(k-7)} \Rightarrow \alpha k(k-7) = 2^2\cdot 3^2\cdot 5\cdot 7^2\cdot 23.$$

   The shortest trip corresponds to the largest k, hence the smallest α.

   We consider two cases: $7 \mid k$ and $7 \nmid k$. If $7 \mid k$, write $k = 7k'$. The above equation implies

$$\alpha k'(k'-1) = 2^2\cdot 3^2\cdot 5\cdot 23.$$

   If $\alpha = 1$, there are no solutions because $2^2\cdot 3^2\cdot 5\cdot 23$ cannot be written in the form $k'(k'-1)$. This can be seen, for example, by considering the member of the factor pair divisible by 23: in $23\cdot 180$ and $46\cdot 90$ the second factor is too large, and in $69\cdot 60$ onwards the second factor is too small.

   If $\alpha = 2$, we find the solution $k' = 46$, corresponding to $k = 7\cdot 46 = 322$. This is a solution to the original problem, as both $\frac{483\cdot 60}{322} = 90$ and $\frac{483\cdot 60}{322-7} = 92$ are integers. This solution minimizes α when $7 \mid k$.

   If $7 \nmid k$, then $7 \nmid k-7$, so $49 \mid \alpha$. Thus $\alpha \geq 49$, and all solutions in this case have larger α than the solution found above.

   Therefore, the shortest trip has $k = 322$. In this trip, Kelvin travels for

$$\frac{483\cdot 60}{322} + \frac{483\cdot 60}{322-7} = 90 + 92 = 182$$

   minutes.

15. Find the greatest prime factor of $2^{56} + (2^{15}+1)(2^{29}+2^{15}+1)$.

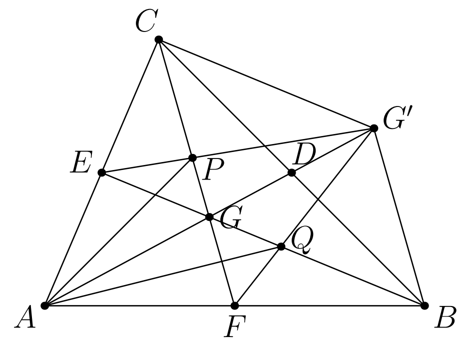
   Proposed by: Kevin Ren

   Answer: $\boxed{113}$

   Expanding the expression gives

$$\begin{aligned} 2^{56}+2^{44}+2^{30}+2^{29}+2^{16}+1 &= 2^{56}+4\cdot 2^{42}+6\cdot 2^{28}+4\cdot 2^{16}+1 \\ &= (2^{14}+1)^4 \\ &= \left((2^7+1)^2-2^8\right)^4 \\ &= (2^7-2^4+1)^4(2^7+2^4+1)^4 \\ &= 113^4\cdot 145^4. \end{aligned}$$

16. Triangle ABC has $BC = 7, CA = 8, AB = 9$. Let D, E, F be the midpoints of BC, CA, AB respectively, and let G be the intersection of AD and BE. G' is the reflection of G across D. Let $G'E$ meet CG at P, and let $G'F$ meet BG at Q. Determine the area of $APG'Q$.



Proposed by: Brice Huang and Kevin Ren

Answer: $\boxed{\frac{16\sqrt{5}}{3}}$

Since G is the centroid of ABC, $AG = 2GD$. Therefore $AG = GG'$, so G is the midpoint of GG'. So, CG and $G'E$ are medians of triangle ACG', and P is its centroid. Similarly, Q is the centroid of ABG'.

This gives $[APG'] = \frac{1}{3}[ACG']$ and $[AQG] = \frac{1}{3}[ABG']$, so $[APGQ'] = \frac{1}{3}[ABG'C]$. Finally, note that $[BCG'] = [BCG] = \frac{1}{3}[ABC]$, so $[ABG'C] = \frac{4}{3}[ABC]$. Therefore $[APGQ'] = \frac{4}{9}[ABC]$.

By Heron's Formula,

$$[ABC] = \sqrt{12 \cdot 5 \cdot 4 \cdot 3} = 12\sqrt{5},$$

so $[APGQ'] = \frac{16\sqrt{5}}{3}$.

17. An *island* is a contiguous set of at least two equal digits. Let $b(n)$ be the number of islands in the binary representation of n. For example, $2020_{10} = 11111100100_2$, so $b(2020) = 3$. Compute

$$b(1) + b(2) + \cdots + b(2^{2020}).$$

Answer: $\boxed{2019 \cdot 2^{2018} + 1}$

Let's first compute the summation when it ranges over all length-2020 bit strings, where two or more consecutive leading zeros count as an island; we will correct for this later.

Let s be a uniformly random length-2020 bit string. $b(s)$ is equal to the number of occurrences of two consecutive equal digits minus the number of occurrences of three consecutive equal digits within the binary representation of n. So, by linearity of expectation,

$$\mathbb{E}[b(s)] = \frac{2019}{2} - \frac{2018}{4} = 505.$$

Therefore,

$$\sum_{s \text{ length } 2020} b(s) = 2^{2020} \cdot 505 = 2^{2018} \cdot 2020.$$

Of course, when $0 \le n < 2^{2018}$ this count includes leading zeros as an island, so the binary string has one more island than the binary number. So,

$$b(0) + b(1) + \cdots + b(2^{2020} - 1) = 2^{2018} \cdot 2020 - 2^{2018} = 2^{2018} \cdot 2019.$$

Finally, we note $b(2^{2020}) = 1$ and $b(0) = 0$, giving a final answer of $2019 \cdot 2^{2018} + 1$.

18. Alice, Bob, Chad, and Denise decide to meet for a virtual group project between 1 and 3 PM, but they don't decide what time. Each of the four group members sign on to Zoom at a uniformly random time between 1 and 2 PM, and they stay for 1 hour. The group gets work done whenever at least three members are present. What is the expected number of minutes that the group gets work done?

Proposed by: Freya Edholm and Kevin Ren

Answer: $\boxed{48}$

Let $x_1 < x_2 < x_3 < x_4$ be the times they arrive. We want the expected value of $60 + x_2 - x_3$. The expected value of x_i is $12i$, so our answer is $60 + 24 - 36 = 48$ minutes.

19. Call a right triangle *peri-prime* if it has relatively prime integer side lengths, perimeter a multiple of 65, and at least one leg with length less than 100. Compute the sum of all possible lengths for the smallest leg of a peri-prime triangle.

Proposed by: Kevin Ren

Answer: $\boxed{241}$

The side lengths of a primitive Pythagorean triple must be a^2-b^2, $2ab$, and a^2+b^2 for positive integers $a > b$ such that $\gcd(a,b) = 1$ and exactly one of a, b is even. Then we must have

$$65 | a^2 - b^2 + 2ab + a^2 + b^2 = 2a(a+b).$$

Thus $65|a(a+b)$. Since $a^2 - b^2 \geq 2a - 1$ and $2ab \geq 2a$, one restriction is $a \leq 50$. Since exactly one of a, b is even, another restriction is that $a + b$ is odd.

We perform casework on how the factors of 65 are distributed among a and $a + b$. Note that $65|a$ is not possible because $a \leq 50$.

Case 1. $65|a + b$.

Since $b < a \leq 65$, $a + b = 65$. Then $2ab > 100$, so we need $a^2 - b^2 = (a + b)(a - b) < 100$. Thus, $(a, b) = (33, 32)$ is the only possibility. The smallest leg is $33^2 - 32^2 = 65$.

Case 2: $5|a$, $13|a + b$, $65 \nmid a + b$.

Then $a + b = 13, 39, 91$. If $a + b = 13$ then $(a, b) = (10, 3)$. The smallest leg is 60. If $a + b = 39$ then $(a, b) = (20, 19), (25, 14), (30, 9), (35, 4)$. Of them, only $(20, 19)$ works. The smallest leg is 39. If $a + b = 91$ then $(a, b) = (50, 41)$, which doesn't work.

Case 3: $13|a$, $5|a + b$, $65 \nmid a + b$.

Then $a = 13, 26, 39$. If $a = 13$ then $b = 2, 12$ work. The smallest legs are 52 and 25, respectively. If $a = 26$, then $b = 9, 19$. None of these work. If $a = 39$, then $b = 6, 16, 26, 36$. None of these work.

The legs we obtained are $25, 39, 52, 60, 65$. Adding these numbers gives 241.

20. Yu Semo and Yu Sejmo have created sequences of symbols $\mathcal{U} = (\mathrm{U}_1, \ldots, \mathrm{U}_6)$ and $\mathcal{J} = (\mathrm{J}_1, \ldots, \mathrm{J}_6)$. These sequences satisfy the following properties.

- Each of the twelve symbols must be Σ, $\#$, $\triangle$, or $\mathbb{Z}$.
- In each of the sets $\{\mathrm{U}_1, \mathrm{U}_2, \mathrm{U}_4, \mathrm{U}_5\}$, $\{\mathrm{J}_1, \mathrm{J}_2, \mathrm{J}_4, \mathrm{J}_5\}$, $\{\mathrm{U}_1, \mathrm{U}_2, \mathrm{U}_3\}$, $\{\mathrm{U}_4, \mathrm{U}_5, \mathrm{U}_6\}$, $\{\mathrm{J}_1, \mathrm{J}_2, \mathrm{J}_3\}$, $\{\mathrm{J}_4, \mathrm{J}_5, \mathrm{J}_6\}$, no two symbols may be the same.
- If integers $d \in \{0, 1\}$ and $i, j \in \{1, 2, 3\}$ satisfy $\mathrm{U}_{i+3d} = \mathrm{J}_{j+3d}$, then $i < j$.

How many possible values are there for the pair $(\mathcal{U}, \mathcal{J})$?

Proposed by: Ankan Bhattacharya

Answer: $\boxed{24}$

First of all, by the second and third conditions, in each of the sets

$$\{\mathrm{J}_1, \mathrm{U}_1, \mathrm{U}_2, \mathrm{U}_3\}, \quad \{\mathrm{J}_1, \mathrm{J}_2, \mathrm{U}_2, \mathrm{U}_3\}, \quad \{\mathrm{J}_1, \mathrm{J}_2, \mathrm{J}_3, \mathrm{U}_3\},$$

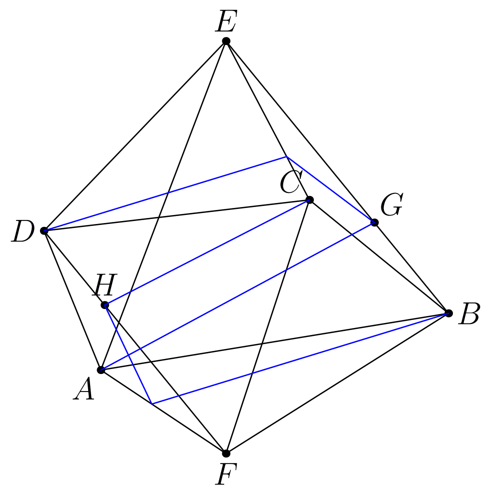
the symbols are all different. It follows that $\mathrm{U}_1 = \mathrm{J}_2$ and $\mathrm{U}_2 = \mathrm{J}_3$, and that J_1 and U_3 are the other two symbols. Analogous reasoning applies to J_4, J_5, J_6, U_4, U_5, U_6.

Now, there are $4! = 24$ ways to select U_1, U_2, U_4, and U_5. We claim that this determines the rest of the symbols. Note that $\mathrm{J}_1 \neq \mathrm{J}_2 = \mathrm{U}_1$, $\mathrm{J}_1 \neq \mathrm{J}_3 = \mathrm{U}_2$, and $\mathrm{J}_1 \neq \mathrm{J}_5 = \mathrm{U}_4$. It follows that $\mathrm{J}_1 = \mathrm{U}_5$; analogously, $\mathrm{J}_4 = \mathrm{U}_2$.

Finally, since U_3 must be different from all of $\mathrm{J}_1 = \mathrm{U}_5$, U_1, and U_2, it must be U_4. Similarly, we obtain $\mathrm{U}_6 = \mathrm{U}_1$.

It is not difficult to check that these sequences satisfy the conditions.

21. Let $ABCDEF$ be a regular octahedron with unit side length, such that $ABCD$ is a square. Points G, H are on segments BE, DF respectively. The planes AGD and BCH divide the octahedron into three pieces, each with equal volume. Compute BG.



Proposed by: Kevin Ren

Answer: $\boxed{\frac{3-\sqrt{19/3}}{2} \text{ (or equivalently, } \frac{9-\sqrt{57}}{6})}$

Let $EG = x$, and let K be on CE such that $EK = EG$. Then pyramid $ADGKE$ has $\frac{2}{3}$ the volume of $ABCDE$. Split $ADGKE$ into tetrahedra $AKEG$ and $AKED$. Note that

$$[AKEG] = x^2[ABCE] = \frac{1}{2}x^2[ABCDE], \qquad [AKED] = x[ADCE] = \frac{1}{2}x[ABCDE].$$

We obtain the equation $\frac{1}{2}(x^2 + x) = \frac{2}{3}$, so $x = \frac{-1+\sqrt{19/3}}{2}$ and $1 - x = \frac{3-\sqrt{19/3}}{2}$.

22. Carol places a king on a 5×5 chessboard. The king starts on the lower-left corner, and each move it steps one square to the right, up, up-right, up-left, or down-right. How many ways are there for the king to get to the top-right corner without visiting the same square twice?

Proposed by: Kevin Ren

Answer: $\boxed{254537}$

Label the chessboard with coordinates, such that the starting square is $(0, 0)$ and the ending square is $(4, 4)$. Let f_i be the number of paths that start at $(0, 0)$ and finish on some square on the diagonal $x + y = i$, such that only the last square of the path lies on this diagonal.

Lemma 1. *For $2 \le i \le 4$, $f_i = 2if_{i-1} + (i-1)f_{i-2}$.*

Proof. Consider the last step on a path counted by f_i. The last step is either orthogonal (up or right) or up-right.

There are $2if_{i-1}$ paths ending in an orthogonal step: $2i$ ways to choose the last step (up or right from one of $(i-1, 0), \ldots, (0, i-1)$), f_{i-1} ways to choose the beginning steps up to the diagonal $x+y = i-1$, and a unique way to connect the last step with the beginning.

There are $(i-1)f_{i-2}$ paths ending in an up-right step: $(i-1)$ ways to choose the last step (up-right from one of $(i-2, 0), \ldots, (0, i-2)$), f_{i-2} ways to choose the beginning steps up to the diagonal $x+y = i-2$, and a unique way to connect. $\square$

Then,

$$\begin{aligned} f_0 &= 1 \\ f_1 &= 2 \\ f_2 &= 9 \\ f_3 &= 58 \\ f_4 &= 491. \end{aligned}$$

Paths from $(0,0)$ to $(4,4)$ either reach some square on the diagonal $x+y=4$ or skip over this diagonal by an up-right step.

There are f_4^2 paths that reach the diagonal $x+y=4$: f_4 ways to choose the beginning steps from $(0,0)$ to the diagonal, f_4 ways to choose the ending steps from $(4,4)$ backward to the diagonal, and a unique way to connect.

There are $4f_3^2$ paths that skip over the diagonal $x+y=4$: f_3 ways to choose the beginning steps to the diagonal $x+y=3$, f_3 ways to choose the ending steps from $(4,4)$ backward to the diagonal $x+y=5$, and 4 ways to connect.

This gives a final answer of $f_4^2 + 4f_3^2 = 254537$.

23. Let f_n be a sequence defined by $f_0 = 2020$ and

$$f_{n+1} = \frac{f_n + 2020}{2020 f_n + 1}$$

for all $n \geq 0$. Determine f_{2020}.

Proposed by: Alexander Katz

Answer: $\boxed{\frac{2021^{2021}+2019^{2021}}{2021^{2021}-2019^{2021}}}$

Pre-solution. Let $r = 2020$. Then

$$f_0 = r = \frac{r}{1},$$

$$f_1 = \frac{2r}{r^2+1},$$

$$f_2 = \frac{r^3+3r}{3r^2+1},$$

$$f_3 = \frac{4r^3+4r}{r^4+6r^2+1}.$$

From this data, we conjecture that

$$f_{i-1} = \frac{(r+1)^i + (r-1)^i}{(r+1)^i - (r-1)^i}$$

for odd i, while the fraction is flipped for even i.

Solution. Let $g_n = \frac{f_n - 1}{f_n + 1}$. Then

$$g_{n+1} = \frac{f_{n+1}-1}{f_{n+1}+1} = \frac{2019 - 2019 f_n}{2021 + 2021 f_n} = -\frac{2019}{2021}\left(\frac{f_n - 1}{f_n + 1}\right) = -\frac{2019}{2021} g_n.$$

As a result,

$$g_{2020} = \left(\frac{2019}{2021}\right)^{2020} g_0$$

But $g_0 = \frac{2020-1}{2020+1} = \frac{2019}{2021}$, so $g_{2020} = \left(\frac{2019}{2021}\right)^{2021}$. Now,

$$\frac{f_{2020}-1}{f_{2020}+1} = g_{2020} \to f_{2020} = \frac{1+g_{2020}}{1-g_{2020}}$$

so that

$$f_{2020} = \frac{2021^{2021}+2019^{2021}}{2021^{2021}-2019^{2021}}$$

24. Farmer John has a 47×53 rectangular square grid. He labels the first row $1, 2, \cdots, 47$, the second row $48, 49, \cdots, 94$, and so on. He plants corn on any square of the form $47x + 53y$, for non-negative integers x, y. Given that the unplanted squares form a contiguous region R, find the perimeter of R.

Proposed by: Kevin Ren

Answer: 794

Let 1 be the lower-left corner and 47 the lower-right corner. The rightmost column, the multiples of 47, is entirely planted. Note that, because $\gcd(47, 53) = 1$, the numbers $53, 53 \cdot 2, \ldots, 53 \cdot 46$ appear in distinct columns. So, one of these appear in each of the leftmost 46 columns, and each of these is the lowest planted square in its column. The unplanted squares are the squares below these numbers.

For $i = 1, \ldots, 46$, let h_i be the number of unplanted squares in column i. The unplanted region R consists of 46 bottom-aligned rectangles with width 1 and height $h_1, \ldots, h_{46}$ placed side by side.

The tops and bottoms of the rectangles each contribute 46 to the perimeter. The left and right boundaries of R contribute h_1 and h_{46}. The boundary between the ith and $i+1$th rectangles contribute $|h_i - h_{i+1}|$. Therefore, the perimeter is

$$2 \cdot 46 + h_1 + h_{46} + \sum_{i=1}^{45} |h_i - h_{i+1}|.$$

Let us determine h_i. If $53k_i$ is the smallest multiple of 53 congruent to $i \pmod{47}$, then $h_i = \frac{53k_i - i}{47}$. Noting that

$$53 \equiv 6 \equiv 8^{-1} \pmod{47},$$

we get $k_i \equiv 8i \pmod{47}$. Thus $k_1 = 8$, yielding $h_1 = \frac{53 \cdot 8 - 1}{47} = 9$. Similarly $k_{46} = 47 - 8 = 39$, yielding $h_{46} = \frac{53 \cdot 39 - 46}{47} = 43$. When we increment i, k_i either increases by 8, in which case

$$h_{i+1} - h_i = \frac{53 \cdot 8 - 1}{47} = 9,$$

or decreases by 39, in which case

$$h_{i+1} - h_i = \frac{53 \cdot (-39) - 1}{47} = -44.$$

The latter case occurs $\lfloor \frac{8 \cdot 46}{47} \rfloor = 7$ times, so the answer is

$$2 \cdot 46 + 9 + 43 + (9 \cdot 38 + 44 \cdot 7) = 794.$$

25. Let $S = \{1, \cdots, 6\}$ and $\mathcal{P}$ be the set of all nonempty subsets of S. Let N equal the number of functions $f : \mathcal{P} \to S$ such that if $A, B \in \mathcal{P}$ are disjoint, then $f(A) \neq f(B)$. Determine the number of positive integer divisors of N.

Proposed by: Ankan Bhattacharya

Answer: 9792

All such functions are of the following type: let σ be any permutation of S and $\varphi : \mathcal{P} \to S$ be a function such that $\varphi(A) \in A$ for all A; then set $f(A) = \sigma(\varphi(A))$ for all A. These clearly work, so we show there are no others.

First of all, $f(\{1\}), \ldots, f(\{6\})$ must be $1, \ldots, 6$ in some order. WLOG assume $f(\{k\}) = k$ for all k.

Now, if $f(A) \notin A$ for some A, then by applying the condition to A and $\{f(A)\}$, we see that $f(A) \neq f(A)$, contradiction. Thus $f(A) \in A$ for all A, giving the characterization above.

Finally, we compute the answer: there are $6! = 2^4 \cdot 3^2 \cdot 5$ ways to choose σ and

$$1^{\binom{6}{1}} 2^{\binom{6}{2}} \cdots 6^{\binom{6}{6}} = 2^{15} \cdot 3^{20} \cdot 4^{15} \cdot 5^6 \cdot 6^1 = 2^{15+2\cdot 15+1} \cdot 3^{20+1} \cdot 5^6 = 2^{46} \cdot 3^{21} \cdot 5^6$$

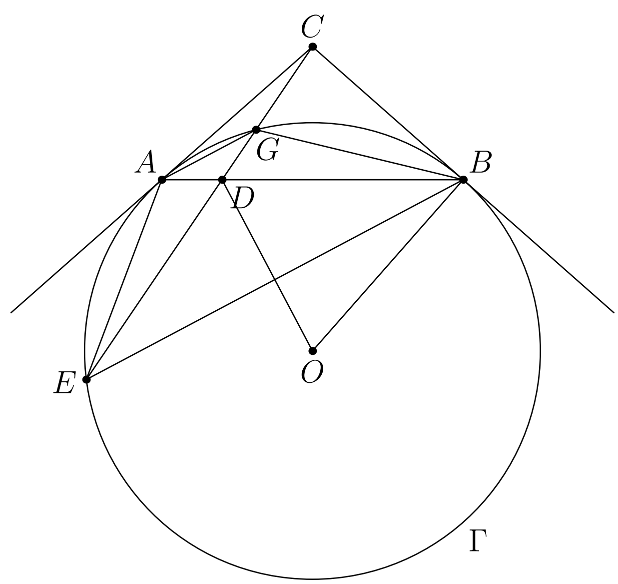
ways to choose φ.

In all, we obtain $N = 2^{50} \cdot 3^{23} \cdot 5^7$ valid functions, so the desired answer is $51 \cdot 24 \cdot 8 = 9792$.

26. Let Γ be a circle centered at O with chord AB. The tangents to Γ at A and B meet at C. A secant from C intersects chord AB at D and Γ at E such that D lies on segment CE. Given that $\angle BOD + \angle EAD = 180°$, $AE = 1$, and $BE = 2$, find CE.

Proposed by: Kevin Ren

Answer: $\boxed{\frac{4\sqrt{2}}{3}}$



Let CD meet Γ at $G \neq E$. Then $\angle EGB = \angle EAB = 180° - \angle BOD$, so $BODG$ is cyclic. Hence, $\angle BDG = \angle BOG$. But, $\angle BOG = \widehat{BG}$ and $\angle BDG = \frac{1}{2}\left(\widehat{BG} + \widehat{AE}\right)$, so $\widehat{BG} = \widehat{AE}$. Hence, $AGBE$ is an isosceles trapezoid. Furthermore, since the tangents at A and B meet on EG, $AGBE$ is also harmonic. Thus, $AE = BG = 1$, $BE = 2$, $AG = \frac{1}{2}$. Now

$$\frac{CG}{CE} = \left(\frac{AG}{AE}\right)^2 = \frac{1}{4}.$$

Furthermore, by Ptolemy's theorem (noting that $GE = AB$),

$$GE^2 = 1 \cdot 1 + \frac{1}{2} \cdot 2 = 2.$$

Hence,

$$CE = \frac{4}{3} GE = \frac{4\sqrt{2}}{3}.$$

27. Let $\phi(n)$ be the number of positive integers less than or equal to n that are relatively prime to n. Evaluate

$$\lim_{m\to\infty} \frac{\sum_{n=1}^{m} \phi(60n)}{\sum_{n=1}^{m} \phi(n)}$$

Proposed by: Alexander Katz

Answer: $\boxed{25}$

Let $g(m) = \frac{3m^2}{\pi^2}$ and $f(m) = \phi(m)$ for convenience. It is well-known that $\lim_{m\to\infty} \frac{1}{g(m)} \sum_{n=1}^{m} f(n) = 1$ (note that we could just as easily choose $g(m) = m^2$; the important part is that $\sum_{n=1}^{m} \phi(n) = \Theta(m^2)$). Let $x = \frac{1}{g(m)} \sum_{n=1}^{m} f(an)$ and $y = \frac{1}{g(m)} \sum_{n=1}^{m} f(\frac{an}{p})$, for some prime $p \mid a$. We claim that

$$x \sum_{n=0}^{\infty} \left(\frac{f(p^n)}{g(p^n)} \right) = y \sum_{n=0}^{\infty} \left(\frac{f(p^{n+1})}{g(p^n)} \right).$$

To demonstrate this, we split the summation in x by $v_p(an)$ in order to utilize the fact that ϕ is generally multiplicative. In particular,

$$\begin{aligned}
xg(m) &= \sum_{n=1}^{m} f(an) \\
&= \sum_{v_p(an)=1} f(an) + \sum_{v_p(an)=2} f(an) + \dots \\
&= f(p) \sum_{v_p(an)=1} f\left(\frac{an}{p}\right) + f(p^2) \sum_{v_p(an)=2} f\left(\frac{an}{p^2}\right) + \dots \\
&= f(p) \sum_{1\le n\le m, v_p(an)=1} f\left(\frac{an}{p}\right) + f(p^2) \sum_{1\le n\le \lfloor \frac{m}{p} \rfloor, v_p(an)=1} \left(\frac{an}{p}\right) + \dots \\
&= f(p) \left(\sum_{1\le n\le m} f\left(\frac{an}{p}\right) - \sum_{1\le n\le \lfloor \frac{m}{p} \rfloor} f(an) \right) + f(p^2) \left(\sum_{1\le n\le \lfloor \frac{m}{p} \rfloor} f\left(\frac{an}{p}\right) - \sum_{1\le n\le \lfloor \frac{m}{p^2} \rfloor} f(an) \right) + \dots \\
&\approx f(p) \left(yg(m) - \frac{xg(m)}{g(p)} \right) + f(p^2) \left(\frac{y}{g(p)} - \frac{x}{g(p^2)} \right) + \dots \\
&= yg(m) \sum_{n=1}^{\infty} \left(\frac{f(p^n)}{g(p^{n-1})} \right) - xg(m) \sum_{n=1}^{\infty} \left(\frac{f(p^n)}{g(p^n)} \right)
\end{aligned}$$

which rearranges to the result (here $\approx$ denotes asymptotic equivalence). Repeating this process, we analogously the more general result

$$x \sum_{n=0}^{\infty} \left(\frac{f(p^n)}{g(p^n)} \right) = y \sum_{n=0}^{\infty} \left(\frac{f(p^{n+v_p(a)})}{g(p^n)} \right).$$

For the prescribed f, g, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{f(p^n)}{g(p^n)} &= \frac{f(1)}{g(1)}+\sum_{n=1}^{\infty}\frac{f(p^n)}{g(p^n)}\\ &= \frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{p^{n-1}(p-1)}{3p^{2n}/pi^2}\\ &= \frac{\pi^2}{3}+\frac{\pi^2(p-1)}{3}\sum_{n=1}^{\infty}\frac{1}{p^{n+1}}\\ &= \frac{\pi^2}{3}+\frac{\pi^2}{3p}\end{aligned}$$

and analogously $\sum_{n=0}^{\infty}\frac{f(p^{n+v_p(a)})}{g(p^n)}=\frac{\pi^2 p^{v_p(a)}}{3}$. Thus,

$$x=\frac{p^{v_p(a)+1}}{p+1}y$$

Finally, with $a=60$ and $p=2,3,5$ respectively, we get

$$\frac{\sum_{n=1}^{m}\phi(60n)}{\sum_{n=1}^{m}\phi(n)}=\frac{2^3}{2+1}\cdot\frac{3^2}{3+1}\cdot\frac{5^2}{5+1}=\boxed{25}$$

28. Call a polynomial f with positive integer coefficients *triangle-compatible* if any three coefficients of f satisfy the triangle inequality. For instance, $3x^3+4x^2+6x+5$ is triangle-compatible, but $3x^3+3x^2+6x+5$ is not. Given that f is a degree 20 triangle-compatible polynomial with -20 as a root, what is the least possible value of $f(1)$?

Proposed by: Kevin Ren

Answer: $\boxed{4641}$

First, consider

$$f(x)=11\left(x^{20}+21x^{19}+21x^{18}+\cdots+21x+21\right)+x+9.$$

Note that f is triangle-compatible and $f(-20)=0$, as required. This f achieves $f(1)=4641$. Let us show this is optimal.

Suppose f be triangle compatible. Divide f by the leading coefficient to get g, a monic polynomial with rational coefficients. Write

$$g(x)=x^{20}+a_{19}x^{19}+a_{18}x^{18}+\cdots+a_1x+a_0.$$

Let $a_k=b_k+21$ for $k=0,1,\ldots,19$. By the Triangle Inequality, $|a_i-a_j|=|b_i-b_j|<1$ for all i,j. Then,

$$g(x)=\left[x^{20}+21x^{19}+\cdots+21x+20\right]+1+\left[b_{19}x^{19}+b_{18}x^{18}+\cdots+b_1x+b_0\right].$$

The first term has -20 as a root, so $g(-20)=0$ is equivalent to

$$1=20^{19}b_{19}-20^{18}b_{18}+\cdots+20b_1-b_0. \tag{1}$$

Let k be the smallest number such that $c_i=kb_i$ is an integer for all $i=0,\ldots,19$. $|b_i-b_j|<1$ implies $|c_i-c_j|\le k-1$ for all i,j.

Lemma 1. *If $k\le 11$, then $c_2=c_3=\cdots=c_{19}=0$.*

Proof. Let i be the largest index such that $c_i \neq 0$. Note that

$$k = 20^{19}c_{19} - 20^{18}c_{18} + \cdots + 20c_1 - c_0,$$

and $|c_j| \le |c_i| + k - 1$ for all $j < i$, while $c_j = 0$ for all $j > i$. Therefore,

$$20^i|c_i| \le 20^{i-1}|c_{i-1}| + \cdots + 20|c_1| + |c_0| + k \le \left(20^{i-1} + \cdots + 20 + 1\right)(|c_i| + k - 1) + k.$$

This rearranges to

$$|c_i| \le \frac{(k-1)\left(20^i + 18\right) + 19}{(18 \cdot 20^i + 1)} \le \frac{10\left(20^i + 18\right) + 19}{(18 \cdot 20^i + 1)}.$$

For $i \ge 2$, the right-hand side is less than 1, contradiction. □

When $k \le 11$, Lemma 1 implies $c_2 = c_3 = \cdots = c_{19} = 0$, so $k = 20c_1 - c_0$. By the Triangle Inequality, $|c_0| = |c_0 - c_{19}| \le k - 1$, so $20|c_1| = |k + c_0| \le 2k - 1$. For $k \le 10$, this implies $c_1 = 0$, so $c_0 = -k$, which is a contradiction. So, there are no solutions with $k \le 10$. If $k = 11$, the only solution is $c_0 = 9, c_1 = 1$, which corresponds to the construction above.

We finish by showing $k \ge 12$ does worse.

Lemma 2. $b_{19} > -\frac{1}{2}$.

Proof. If $b_{19} \le -\frac{1}{2}$, $|b_i - b_{19}| < 1$ implies $-\frac{3}{2} < b_i < \frac{1}{2}$ for each $0 \le i \le 18$. Then, (1) implies

$$\frac{1}{2} \cdot 20^{19} \le 20^{19}|b_{19}| \le 20^{18}|b_{18}| + 20^{17}|b_{17}| + \cdots + 20^1|b_1| + |b_0| + 1 < \frac{3}{2}\left(20^{18} + \cdots + 20^1 + 1\right) + 1.$$

This is a contradiction. □

Thus, $b_{19} \ge -\frac{1}{2}$, so $b_i \ge -\frac{3}{2}$ for each $0 \le i \le 18$. If $k \ge 12$, then

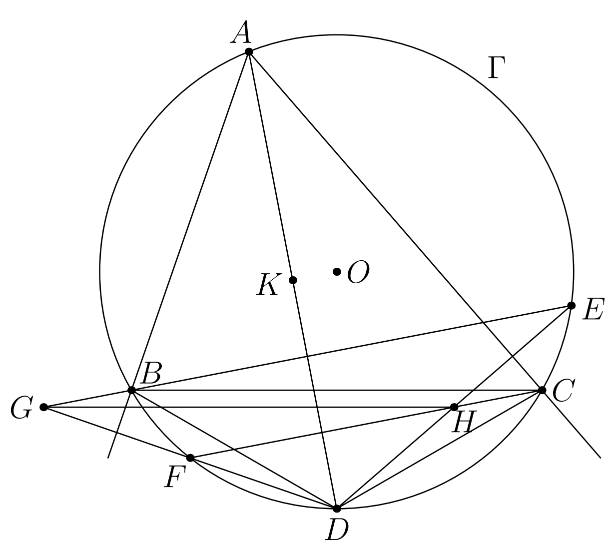
$$P(1) \ge 12\left(1 + \left(21 - \frac{1}{2}\right) + 19\left(21 - \frac{3}{2}\right)\right) = 4704 > 4641.$$

Therefore, 4641 is minimal.

29. Let ABC be a triangle with circumcircle Γ and let D be the midpoint of minor arc BC. Let E, F be on Γ such that $DE \perp AC$ and $DF \perp AB$. Lines BE and DF meet at G, and lines CF and DE meet at H. Given that $AB = 8, AC = 10$, and $\angle BAC = 60°$, find the area of $BCHG$.

Proposed by: Kevin Ren

Answer: $\boxed{2\sqrt{3}}$



Note that

$$\angle(AD, BE) = \frac{\widehat{BD} + \widehat{AE}}{2} = \frac{\widehat{CD} + \widehat{AE}}{2} = \angle(AC, DE) = 90^\circ.$$

Therefore, $BE \perp AD$. Also $DF \perp AB$, so G is the orthocenter of ABD. Similarly, H is the orthocenter of ACD.

Let O be the circumcenter of ABC, and identify O with the zero vector. We have $\vec{G} = \vec{A} + \vec{B} + \vec{D}$ and $\vec{H} = \vec{A} + \vec{C} + \vec{D}$, so $\vec{G} - \vec{H} = \vec{B} - \vec{C}$, which means $BCHG$ is a parallelogram.

Let K be the midpoint of AD. Then $\vec{G} - \vec{B} = \vec{A} + \vec{D} = 2\vec{K}$, so $BG = 2KO$. Moreover,

$$\angle EBC = \angle EDC = 90^\circ - \angle ACD = 90^\circ - \angle DOK.$$

So,

$$\sin \angle GBC = \sin \angle EBC = \cos \angle DOK = \frac{KO}{R}.$$

Note that the circumradius is $R = \frac{BC}{\sqrt{3}}$. Hence,

$$\begin{aligned} [BCHG] &= BG \cdot BC \cdot \sin \angle GBC = 2KO \cdot BC \cdot \frac{KO}{R} \\ &= 2\sqrt{3}KO^2 = 2\sqrt{3}\left(R^2 - \frac{AD^2}{4}\right). \end{aligned}$$

Now we compute by Ptolemy's theorem,

$$AD = \frac{AB \cdot CD + AC \cdot BD}{BC} = \frac{AB + AC}{\sqrt{3}} = 6\sqrt{3},$$

noting that $\frac{CD}{BC} = \frac{BD}{BC} = \frac{1}{\sqrt{3}}$. By Law of Cosines,

$$BC^2 = 8^2 + 10^2 - 8 \cdot 10 = 84 \implies R^2 = 28.$$

Thus,

$$[BCHG] = 2\sqrt{3}(28 - 27) = 2\sqrt{3}.$$

30. For a positive integer n, let $\Omega(n)$ denote the number of prime factors of n, counting multiplicity. Let $f_1(n)$ and $f_3(n)$ denote the sum of positive divisors $d|n$ where $\Omega(d) \equiv 1 \pmod 4$ and $\Omega(d) \equiv 3 \pmod 4$, respectively. For example, $f_1(72) = 72+2+3 = 77$ and $f_3(72) = 8+12+18 = 38$. Determine $f_3(6^{2020}) - f_1(6^{2020})$.

Proposed by: Brice Huang

Answer: $\boxed{\frac{6^{2021}-3^{2021}-2^{2021}-1}{10}}$

Let us analogously define $f_0(n)$ and $f_2(n)$ as, respectively, the sum of divisors $d|n$ where $\Omega(d) \equiv 0 \pmod 4$ and $\Omega(d) \equiv 2 \pmod 4$. Moreover, let us define

$$g(n) = \sum_{d|n} d i^{\Omega(d)} = f_0(n) + i f_1(n) - f_2(n) - i f_3(n),$$

where $i = \sqrt{-1}$. The answer is $-\mathrm{Im}\left(g(6^{2020})\right)$.

Note that g is multiplicative: if m, n are relatively prime, then

$$g(m)g(n) = \sum_{d_1|m} d_1 i^{\Omega(d_1)} \sum_{d_2|n} d_2 i^{\Omega(d_2)} = \sum_{d_1|m, d_2|n} d_1 d_2 i^{\Omega(d_1 d_2)} = \sum_{d|mn} d i^{\Omega(d)} = g(mn).$$

Moreover, the evaluations of g on prime powers p^k are

$$g(p^k) = 1 + pi + (pi)^2 + \cdots + (pi)^k = \frac{1 - (pi)^{k+1}}{1 - pi}.$$

Thus,

$$\begin{aligned} g(6^{2020}) &= g(2^{2020})g(3^{2020}) \\ &= \frac{1-2^{2021}i}{1-2i} \cdot \frac{1-3^{2021}i}{1-3i} \\ &= \frac{(1-6^{2021})-(2^{2021}+3^{2021})i}{-5-5i} \\ &= \frac{(6^{2021}+3^{2021}+2^{2021}-1)-(6^{2021}-3^{2021}-2^{2021}-1)i}{10}. \end{aligned}$$

This gives an answer of

$$-\operatorname{Im}\left(g(6^{2020})\right) = \frac{6^{2021}-3^{2021}-2^{2021}-1}{10}.$$