National Championship 2020

Challenger / Open Division

April 26 — May 9, 2020

Time Limit: 3 hours for all teams in the Challenger Division, and for teams in the Open Division with more than four members. 4 hours for teams in the Open Division with four or fewer members.
Each problem is worth 1 point.

1. If U, S, M, C, A are distinct (not necessarily positive) integers such that $U \cdot S \cdot M \cdot C \cdot A = 2020$, what is the greatest possible value of $U + S + M + C + A$?

2. Sarah is fighting a dragon in DnD. She rolls two fair twenty-sided dice numbered $1, 2, \ldots, 20$. She vanquishes the dragon if the product of her two rolls is a multiple of 4. What is the probability that the dragon is vanquished?

3. If $x(y+1) = 41$ and $x^2(y^2+1) = 881$, determine all possible pairs of real numbers (x, y).

4. Let $ABCDEF$ be a regular hexagon with side length two. Extend FE and BD to meet at G. Compute the area of $ABGF$.

5. Call a positive integer n an $A - B$ number if the base A and base B representations of n are three-digit numbers that are reverses of each other. For example, 87 is a $5 - 6$ number because $87 = 223_6 = 322_5$. Compute the sum of all $7 - 11$ numbers.

6. Alex is thinking of a number that is divisible by all of the positive integers 1 through 200 inclusive except for two consecutive numbers. What is the smaller of these numbers?

7. Jenn is competing in a puzzle hunt with six regular puzzles and one additional meta-puzzle. Jenn can solve any puzzle regularly. Additionally, if she has already solved the meta-puzzle, Jenn can also back-solve a puzzle. A back-solve is distinguishable from a regular solve. The meta puzzle cannot be the first puzzle solved. How many possible solve orders for the seven puzzles are possible?

   For example, Jenn may solve #3, solve #5, solve #6, solve the meta-puzzle, solve #2, solve #1, and then solve #4. However, she may not solve #2, solve #4, solve #6, back-solve #1, solve #3, solve #5, and then solve the meta-puzzle.

8. Two altitudes of a triangle have lengths 8 and 15. How many possible integer lengths are there for the third altitude?

9. Let Ω be a unit circle and A be a point on Ω. An angle $0 < \theta < 180^\circ$ is chosen uniformly at random, and Ω is rotated θ degrees clockwise about A. What is the expected area swept by this rotation?

10. If $0 < x < \frac{\pi}{2}$ and $\frac{\sin x}{1+\cos x} = \frac{1}{3}$, what is $\frac{\sin 2x}{1+\cos 2x}$?

11. A permutation of $USMCAUSMCA$ is selected uniformly at random. What is the probability that this permutation is exactly one transposition away from $USMCAUSMCA$ (i.e. does not equal $USMCAUSMCA$, but can be turned into $USMCAUSMCA$ by swapping one pair of letters)?

12. Let a, b, c, d be the roots of the quartic polynomial $f(x) = x^4 + 2x + 4$. Find the value of
$$\frac{a^2}{a^3+2} + \frac{b^2}{b^3+2} + \frac{c^2}{c^3+2} + \frac{d^2}{d^3+2}.$$

13. Equiangular octagon $ABCDEFGH$ is inscribed in a circle centered at O. Chords AD and BG intersect at K. Given that $AB = 2$ and the octagon has area 15, compute the area of $HAKBO$.

14. Kelvin the Nanofrog is visiting his friend, Alex the Nanokat, who lives 483 nanometers away. On his trip to Alex's home, Kelvin travels at k nanometers an hour, where k is an integer, and completes the trip in an integer number of minutes. On his return journey, he travels slower by 7 nanometers an hour, and completes the trip in an integer number of minutes. What is the smallest total number of minutes Kelvin could have spent traveling?

15. Find the greatest prime factor of $2^{56} + (2^{15} + 1)(2^{29} + 2^{15} + 1)$.

16. Triangle ABC has $BC = 7, CA = 8, AB = 9$. Let D, E, F be the midpoints of BC, CA, AB respectively, and let G be the intersection of AD and BE. G' is the reflection of G across D. Let $G'E$ meet CG at P, and let $G'F$ meet BG at Q. Determine the area of $APG'Q$.

17. An *island* is a contiguous set of at least two equal digits. Let $b(n)$ be the number of islands in the binary representation of n. For example, $2020_{10} = 11111100100_2$, so $b(2020) = 3$. Compute

$$b(1) + b(2) + \cdots + b(2^{2020}).$$

18. Alice, Bob, Chad, and Denise decide to meet for a virtual group project between 1 and 3 PM, but they don't decide what time. Each of the four group members sign on to Zoom at a uniformly random time between 1 and 2 PM, and they stay for 1 hour. The group gets work done whenever at least three members are present. What is the expected number of minutes that the group gets work done?

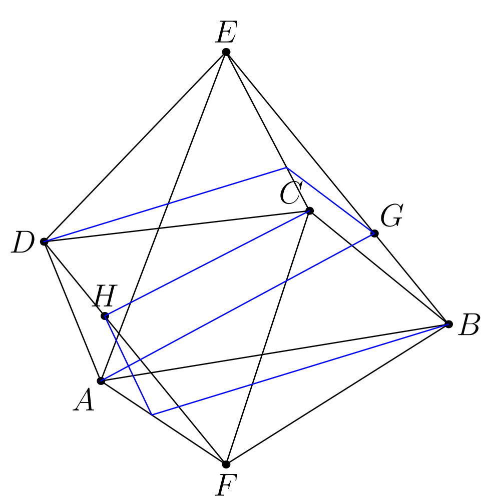
19. Call a right triangle *peri-prime* if it has relatively prime integer side lengths, perimeter a multiple of 65, and at least one leg with length less than 100. Compute the sum of all possible lengths for the smallest leg of a peri-prime triangle.

20. Yu Semo and Yu Sejmo have created sequences of symbols $\mathcal{U} = (\mathrm{U}_1, \ldots, \mathrm{U}_6)$ and $\mathcal{J} = (\mathrm{J}_1, \ldots, \mathrm{J}_6)$. These sequences satisfy the following properties.

    - Each of the twelve symbols must be Σ, $\#$, $\triangle$, or $\mathbb{Z}$.
    - In each of the sets $\{\mathrm{U}_1, \mathrm{U}_2, \mathrm{U}_4, \mathrm{U}_5\}$, $\{\mathrm{J}_1, \mathrm{J}_2, \mathrm{J}_4, \mathrm{J}_5\}$, $\{\mathrm{U}_1, \mathrm{U}_2, \mathrm{U}_3\}$, $\{\mathrm{U}_4, \mathrm{U}_5, \mathrm{U}_6\}$, $\{\mathrm{J}_1, \mathrm{J}_2, \mathrm{J}_3\}$, $\{\mathrm{J}_4, \mathrm{J}_5, \mathrm{J}_6\}$, no two symbols may be the same.
    - If integers $d \in \{0, 1\}$ and $i, j \in \{1, 2, 3\}$ satisfy $\mathrm{U}_{i+3d} = \mathrm{J}_{j+3d}$, then $i < j$.

    How many possible values are there for the pair $(\mathcal{U}, \mathcal{J})$?

21. Let $ABCDEF$ be a regular octahedron with unit side length, such that $ABCD$ is a square. Points G, H are on segments BE, DF respectively. The planes AGD and BCH divide the octahedron into three pieces, each with equal volume. Compute BG.



22. Carol places a king on a 5×5 chessboard. The king starts on the lower-left corner, and each move it steps one square to the right, up, up-right, up-left, or down-right. How many ways are there for the king to get to the top-right corner without visiting the same square twice?

23. Let f_n be a sequence defined by $f_0 = 2020$ and

$$f_{n+1} = \frac{f_n + 2020}{2020 f_n + 1}$$

for all $n \geq 0$. Determine f_{2020}.

24. Farmer John has a 47×53 rectangular square grid. He labels the first row $1, 2, \cdots, 47$, the second row $48, 49, \cdots, 94$, and so on. He plants corn on any square of the form $47x + 53y$, for non-negative integers x, y. Given that the unplanted squares form a contiguous region R, find the perimeter of R.

25. Let $S = \{1, \cdots, 6\}$ and $\mathcal{P}$ be the set of all nonempty subsets of S. Let N equal the number of functions $f : \mathcal{P} \to S$ such that if $A, B \in \mathcal{P}$ are disjoint, then $f(A) \neq f(B)$. Determine the number of positive integer divisors of N.

26. Let Γ be a circle centered at O with chord AB. The tangents to Γ at A and B meet at C. A secant from C intersects chord AB at D and Γ at E such that D lies on segment CE. Given that $\angle BOD + \angle EAD = 180°$, $AE = 1$, and $BE = 2$, find CE.

27. Let $\phi(n)$ be the number of positive integers less than or equal to n that are relatively prime to n. Evaluate

$$\lim_{m \to \infty} \frac{\sum_{n=1}^{m} \phi(60n)}{\sum_{n=1}^{m} \phi(n)}$$

28. Call a polynomial f with positive integer coefficients *triangle-compatible* if any three coefficients of f satisfy the triangle inequality. For instance, $3x^3 + 4x^2 + 6x + 5$ is triangle-compatible, but $3x^3 + 3x^2 + 6x + 5$ is not. Given that f is a degree 20 triangle-compatible polynomial with -20 as a root, what is the least possible value of $f(1)$?

29. Let ABC be a triangle with circumcircle Γ and let D be the midpoint of minor arc BC. Let E, F be on Γ such that $DE \perp AC$ and $DF \perp AB$. Lines BE and DF meet at G, and lines CF and DE meet at H. Given that $AB = 8, AC = 10$, and $\angle BAC = 60°$, find the area of $BCHG$.

30. For a positive integer n, let $\Omega(n)$ denote the number of prime factors of n, counting multiplicity. Let $f_1(n)$ and $f_3(n)$ denote the sum of positive divisors $d|n$ where $\Omega(d) \equiv 1 \pmod 4$ and $\Omega(d) \equiv 3 \pmod 4$, respectively. For example, $f_1(72) = 72 + 2 + 3 = 77$ and $f_3(72) = 8 + 12 + 18 = 38$. Determine $f_3(6^{2020}) - f_1(6^{2020})$.