National Finals 2019 – Premier Division

April 28 — May 4, 2019

Time Limit: 2 hours.
Each problem is worth 7 points.

1. Kelvin the Frog and Alex the Kat are playing a game on an initially empty blackboard. Kelvin begins by writing a digit. Then, the players alternate inserting a digit anywhere into the number currently on the blackboard, including possibly a leading zero (e.g. 12 can become 123, 142, 512, 012, etc.). Alex wins if the blackboard shows a perfect square at any time, and Kelvin's goal is prevent Alex from winning. Does Alex have a winning strategy?

2. Let $n \geq 2$ be an even integer. Find the maximum integer k (in terms of n) such that 2^k divides $\binom{n}{m}$ for some $0 \leq m \leq n$.

3. Let ABC be a scalene triangle. The incircle of ABC touches $\overline{BC}$ at D. Let P be a point on $\overline{BC}$ satisfying $\angle BAP = \angle CAP$, and M be the midpoint of $\overline{BC}$. Define Q to be on $\overline{AM}$ such that $\overline{PQ} \perp \overline{AM}$. Prove that the circumcircle of $\triangle AQD$ is tangent to $\overline{BC}$.

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(f(x) + y)^2 = (x - y)(f(x) - f(y)) + 4f(x)f(y).$$

5. The number 2019 is written on a blackboard. Every minute, if the number a is written on the board, Evan erases it and replaces it with a number chosen from the set

$$\{0, 1, 2, \dots, \lceil 2.01a \rceil\}$$

uniformly at random. Is there an integer N such that the board reads 0 after N steps with at least 99% probability?

6. A *mirrored polynomial* is a polynomial f with real coefficients and degree 100 such that the x^{50} coefficient of f is 1, and $f(x) = x^{100} f(1/x)$ holds for all positive real numbers x. Find the smallest real constant C such that any mirrored polynomial f satisfying $f(1) \geq C$ has a complex root z obeying $|z| = 1$.

7. Let $AXBY$ be a convex quadrilateral. The incircle of $\triangle AXY$ has center I_A and touches $\overline{AX}$ and $\overline{AY}$ at A_1 and A_2 respectively. The incircle of $\triangle BXY$ has center I_B and touches $\overline{BX}$ and $\overline{BY}$ at B_1 and B_2 respectively. Define $P = \overline{XI_A} \cap \overline{YI_B}$, $Q = \overline{XI_B} \cap \overline{YI_A}$, and $R = \overline{A_1B_1} \cap \overline{A_2B_2}$.

   a. Prove that if $\angle AXB = \angle AYB$, then P, Q, R are collinear.

   b. Prove that if there exists a circle tangent to all four sides of $AXBY$, then P, Q, R are collinear.

8. Find all pairs of positive integers (m, n) such that $(2^m - 1)(2^n - 1)$ is a perfect square.